

AN EXPLICIT VOLUME FORMULA FOR THE LINK $7_3^2(\alpha, \alpha)$ CONE-MANIFOLDS

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ABSTRACT. We calculate the volume of the 7_3^2 link cone-manifolds using the Schläfli formula. As an application, we give the volume of the cyclic coverings branched over the link.

1. INTRODUCTION

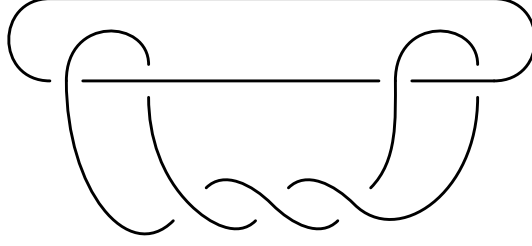
Let us denote the link complement of 7_3^2 in Rolfsen's link table by X . Note that it is a hyperbolic knot. Hence by Mostow-Prasad rigidity theorem, X has a unique hyperbolic structure. Let ρ_∞ be the holonomy representation from $\pi_1(X)$ to $\mathrm{PSL}(2, \mathbb{C})$ and denote $\rho_\infty(\pi_1(X))$ by Γ , a Kleinian group. X is a $(\mathrm{PSL}(2, \mathbb{C}), \mathbb{H}^3)$ -manifold and can be identified with \mathbb{H}^3/Γ . Thurston's orbifold theorem guarantees an orbifold, $X(\alpha) = X(\alpha, \alpha)$, with underlying space S^3 and with the link 7_3^2 as the singular locus of the cone-angle $\alpha = 2\pi/k$ for some nonzero integer k , can be identified with \mathbb{H}^3/Γ' for some $\Gamma' \in \mathrm{PSL}(2, \mathbb{C})$; the hyperbolic structure of X is deformed to the hyperbolic structure of $X(\alpha)$. For the intermediate angles whose multiples are not 2π and not bigger than π , Kojima [10] showed that the hyperbolic structure of $X(\alpha)$ can be obtained uniquely by deforming nearby orbifold structures. Note that there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ for the link 7_3^2 such that $X(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi]$ [19, 8, 10, 20]. For further knowledge of cone-manifolds a reader can consult [1, 7].

Even though we have wide discussions on orbifolds, it seems to us we have a little in regard to cone-manifolds. Explicit volume formulae for hyperbolic cone-manifolds of knots and links are known a little. The volume formulae for hyperbolic cone-manifolds of the knot 4_1 [8, 10, 11, 15], the knot 5_2 [13], the link 5_1^2 [16], the link 6_2^2 [17], and the link 6_3^2 [2] have been computed. In [9] a method of calculating the volumes of two-bridge knot cone-manifolds was introduced but without explicit formulae. In [7, 6], explicit volume formulae of cone-manifolds for the hyperbolic twist knot and for the knot with Conway notation $C(2n, 3)$ are computed. Similar methods are used for computing Chern-Simons invariants of orbifolds for the twist knot and $C(2n, 3)$ knot in [5, 4].

The main purpose of the paper is to find an explicit and efficient volume formula of hyperbolic cone-manifolds for the link 7_3^2 . The following theorem gives the volume formula for $X(\alpha)$.

Theorem 1.1. *Let $X(\alpha)$, $0 \leq \alpha < \alpha_0$ be the hyperbolic cone-manifold with underlying space S^3 and with singular set the link 7_3^2 of cone-angle α . $X(0)$ denotes X . Then the volume of $X(\alpha)$ is given by the following formula*

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FIGURE 1. Link 7_3^2 in Rolfsen's link table

$$\text{Vol}(X(\alpha)) = \int_{\alpha}^{\pi} 2 \log \left| \frac{A - iV}{A + iV} \right| d\alpha,$$

where for $A = \cot \frac{\alpha}{2}$, V ($\text{Re}(V) \leq 0$ and $\text{Im}(V) \geq 0$ is the largest) is a zero of the Riley-Mednykh polynomial $P = P(V, A)$ for the link 7_3^2 given below.

$$\begin{aligned} P = & 8V^5 + 8A^2V^4 + (8A^4 + 16A^2 - 8)V^3 + (4A^6 + 8A^4 - 12A^2)V^2 \\ & + (A^8 + 4A^6 - 2A^4 - 12A^2 + 1)V - 4A^6 - 8A^4 + 4A^2. \end{aligned}$$

The following corollary gives the hyperbolic volume of the k -fold strictly-cyclic covering [12, 18] over the link 7_3^2 , $M_k(X)$, for $k \geq 3$.

Corollary 1.2. *The volume of $M_k(X)$ is given by the following formula*

$$\text{Vol}(M_k(X)) = k \int_{\frac{2\pi}{k}}^{\pi} 2 \log \left| \frac{A - iV}{A + iV} \right| d\alpha,$$

where for $A = \cot \frac{\alpha}{2}$, V ($\text{Re}(V) \leq 0$ and $\text{Im}(V) \geq 0$ is the largest) is a zero of the Riley-Mednykh polynomial $P = P(V, A)$ for the link 7_3^2 .

In Section 2, we present the fundamental group $\pi_1(X)$ of X with slope $9/16$. In Section 3, we give the defining equation of the representation variety of $\pi_1(X)$. In Section 4, we compute the longitude of the link 7_3^2 using the Pythagorean theorem. And in Section 5, we give the proof of Theorem 1.1 using the Schläfli formula.

2. LINK 7_3^2

Link 7_3^2 is presented in Figure 1. It is the same as W_3 from [2]. The slope of this link is $7/16$. The link with slope $9/16$ is the mirror of the link 7_3^2 . Since the volume of the link with slope $7/16$ is the same as the volume of link with slope $9/16$, in the rest of the paper, the link with slope $9/16$ is used.

The following fundamental group of X is stated in [2] with slope $7/16$.

Proposition 2.1.

$$\pi_1(X) = \langle s, t \mid sws^{-1}w^{-1} = 1 \rangle,$$

where $w = s^{-1}[s, t]^2[s, t^{-1}]^2$.

3. $(\mathrm{PSL}(2, \mathbb{C}), \mathbb{H}^3)$ STRUCTURE OF $X(\alpha)$

Let $R = \mathrm{Hom}(\pi_1(X), \mathrm{SL}(2, \mathbb{C}))$. Given a set of generators, s, t , of the fundamental group for $\pi_1(X)$, we define a set $R(\pi_1(X)) \subset \mathrm{SL}(2, \mathbb{C})^2 \subset \mathbb{C}^8$ to be the set of all points $(h(s), h(t))$, where h is a representation of $\pi_1(X)$ into $\mathrm{SL}(2, \mathbb{C})$. Since the defining relation of $\pi_1(X)$ gives the defining equation of $R(\pi_1(X))$ [21], $R(\pi_1(X))$ is an affine algebraic set in \mathbb{C}^8 . $R(\pi_1(X))$ is well-defined up to isomorphisms which arise from changing the set of generators. We say elements in R which differ by conjugations in $\mathrm{SL}(2, \mathbb{C})$ are *equivalent*. A point on the variety gives the $(\mathrm{PSL}(2, \mathbb{C}), \mathbb{H}^3)$ structure of $X(\alpha)$.

Let

$$h(s) = \begin{bmatrix} \cos \frac{\alpha}{2} & ie^{\frac{\rho}{2}} \sin \frac{\alpha}{2} \\ ie^{-\frac{\rho}{2}} \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{bmatrix}, \quad h(t) = \begin{bmatrix} \cos \frac{\alpha}{2} & ie^{-\frac{\rho}{2}} \sin \frac{\alpha}{2} \\ ie^{\frac{\rho}{2}} \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{bmatrix}.$$

Then h becomes a representation if and only if $A = \cot \frac{\alpha}{2}$ and $V = \cosh \rho$ satisfies a polynomial equation [21, 14]. We call the defining polynomial of the algebraic set $\{(V, A)\}$ as the *Riley-Mednykh polynomial* for the link 7_3^2 . Throughout the paper, h can be sometimes any representation and sometimes the unique hyperbolic representation.

Given the fundamental group of X ,

$$\pi_1(X) = \langle s, t \mid sws^{-1}w^{-1} = 1 \rangle,$$

where $w = s^{-1}[s, t]^2[s, t^{-1}]^2$, let $S = h(s)$, $T = h(t)$ and $W = h(w)$. Then the trace of S and the trace of T are both $2 \cos \frac{\alpha}{2}$.

Lemma 3.1. *For $n \in \mathrm{SL}(2, \mathbb{C})$ which satisfies $nS = S^{-1}n$, $nT = T^{-1}n$, and $n^2 = -I$,*

$$SWS^{-1}W^{-1} = -(SWn)^2.$$

Proof.

$$\begin{aligned} (SWn)^2 &= SWnSWn = SWS^{-1}n(S^{-1}(STS^{-1}T^{-1})^2(ST^{-1}S^{-1}T)^2)n \\ &= SWS^{-1}(S(S^{-1}T^{-1}ST)^2(S^{-1}TST^{-1})^2)n^2 = -SWS^{-1}W^{-1}. \end{aligned}$$

□

From the structure of the algebraic set of $R(\pi_1(X))$ with coordinates $h(s)$ and $h(t)$ we have the defining equation of $R(\pi_1(X))$. The following theorem is stated in [2, Proposition 4] with slope $7/16$.

Theorem 3.2. *h is a representation of $\pi_1(X)$ if V is a root of the following Riley-Mednykh polynomial $P = P(V, A)$ which is given below.*

$$\begin{aligned} P &= 8V^5 + 8A^2V^4 + (8A^4 + 16A^2 - 8)V^3 + (4A^6 + 8A^4 - 12A^2)V^2 \\ &\quad + (A^8 + 4A^6 - 2A^4 - 12A^2 + 1)V - 4A^6 - 8A^4 + 4A^2. \end{aligned}$$

Proof. Note that $SWS^{-1}W^{-1} = I$, which gives the defining equations of $R(\pi_1(X))$, is equivalent to $(SWn)^2 = -I$ in $\mathrm{SL}(2, \mathbb{C})$ by Lemma 3.1 and $(SWn)^2 = -I$ in $\mathrm{SL}(2, \mathbb{C})$ is equivalent to $\mathrm{tr}(SWn) = 0$.

We can find two n 's in $\mathrm{SL}(2, \mathbb{C})$ which satisfies $nS = S^{-1}n$ and $n^2 = -I$ by direct computations. The existence and the uniqueness of the isometry (the involution) which is represented by n are shown in [3, p. 46]. Since two n 's give the same element in $\mathrm{PSL}(2, \mathbb{C})$, we use one of them. Hence, we may assume

$$n = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

$$S = \begin{bmatrix} \cos \frac{\alpha}{2} & ie^{\frac{\rho}{2}} \sin \frac{\alpha}{2} \\ ie^{-\frac{\rho}{2}} \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{bmatrix}, \quad T = \begin{bmatrix} \cos \frac{\alpha}{2} & ie^{-\frac{\rho}{2}} \sin \frac{\alpha}{2} \\ ie^{\frac{\rho}{2}} \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{bmatrix}.$$

Recall that P is the defining polynomial of the algebraic set $\{(V, A)\}$ and the defining polynomial of $R(\pi_1(X))$ corresponding to our choice of $h(s)$ and $h(t)$. By direct computation P is a factor of $\mathrm{tr}(SWn) = -4i \sinh \rho (2V^2 + A^4 + 2A^2 - 1)P$. As in [2], P can not be $\sinh \rho$ or have only real roots. Also, P can not have only purely imaginary roots similarly. P in the theorem is the only factor of $\mathrm{tr}(SWn)$ which is different from $\sinh \rho$ and has roots which are not real or purely imaginary. P is the Riley-Mednykh polynomial. \square

4. LONGITUDE

Let $l_s = ws$ and $l_t = (t^{-1}[t, s]^2[t, s^{-1}]^2)t$. Then l_s and l_t are the longitudes which are null-homologous in X . Let $L_S = h(l_s)$ and Let $L_T = h(l_t)$.

Lemma 4.1. $\mathrm{tr}(S^{-1}L_T) = \mathrm{tr}(S)$ and $\mathrm{tr}(T^{-1}L_S) = \mathrm{tr}(T)$.

Proof. Since

$$\begin{aligned} S^{-1}L_T &= S^{-1}(T^{-1}(TST^{-1}S^{-1}TST^{-1}S^{-1} \cdot TS^{-1}T^{-1}STS^{-1}T^{-1}S)T) \\ &= (T^{-1}S^{-1}TST^{-1}S^{-1}T)(S^{-1})(T^{-1}S^{-1}TST^{-1}S^{-1}T)^{-1}, \end{aligned}$$

$$\mathrm{tr}(S^{-1}L_T) = \mathrm{tr}(S^{-1}) = \mathrm{tr}(S).$$

The second statement can be obtained in a similar way. \square

Definition. The *complex length* of the longitude l (l_s or l_t) of the link 7_3^2 is the complex number γ_α modulo $4\pi\mathbb{Z}$ satisfying

$$\mathrm{tr}(h(l)) = 2 \cosh \frac{\gamma_\alpha}{2}.$$

Note that $l_\alpha = |\mathrm{Re}(\gamma_\alpha)|$ is the real length of the longitude of the cone-manifold $X(\alpha)$.

By sending common fixed points of T and $L_T = h(l_t)$ to 0 and ∞ , we have

$$T = \begin{bmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{bmatrix}, \quad L_T = \begin{bmatrix} e^{\frac{\gamma_\alpha}{2}} & 0 \\ 0 & e^{-\frac{\gamma_\alpha}{2}} \end{bmatrix},$$

and the following normalized line matrices of T (resp. L_T) which share the fixed points with T (resp. L_T).

$$\begin{aligned}
 l(T) &\equiv \frac{T - T^{-1}}{2i \sinh \frac{i\alpha}{2}} \\
 &= \frac{1}{i(e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}})} \begin{bmatrix} e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} - e^{\frac{i\alpha}{2}} \end{bmatrix} \\
 &= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},
 \end{aligned}$$

$$\begin{aligned}
 l(L_T) &\equiv \frac{L_T - L_T^{-1}}{2i \sinh \frac{\gamma_\alpha}{2}} \\
 &= \frac{1}{i(e^{\frac{\gamma_\alpha}{2}} - e^{-\frac{\gamma_\alpha}{2}})} \begin{bmatrix} e^{\frac{\gamma_\alpha}{2}} - e^{-\frac{\gamma_\alpha}{2}} & 0 \\ 0 & e^{-\frac{\gamma_\alpha}{2}} - e^{\frac{\gamma_\alpha}{2}} \end{bmatrix} \\
 &= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},
 \end{aligned}$$

which give the orientations of axes of T and L_T .

Now, we are ready to prove the following theorem which gives Theorem 4.3. Recall that γ_α modulo $4\pi\mathbb{Z}$ is the *complex length* of the longitude l_s or l_t of $X(\alpha)$. The following theorem is a particular case of Proposition 5 from [2].

Theorem 4.2. (*Pythagorean Theorem*) [2] *Let $X(\alpha)$ be a hyperbolic cone-manifold and let ρ be the complex distance between the oriented axes S and T . Then we have*

$$i \cosh \rho = \cot \frac{\alpha}{2} \coth \left(\frac{\gamma_\alpha}{4} \right).$$

Proof.

$$\begin{aligned}
\cosh \rho &= -\frac{\operatorname{tr}(l(S)l(T))}{2} \\
&= -\frac{\operatorname{tr}(l(S)l(L_T))}{2} \\
&= \frac{\operatorname{tr}((S - S^{-1})(L_T - L_T^{-1}))}{8 \sinh \frac{i\alpha}{2} \sinh \frac{\gamma_\alpha}{2}} \\
&= \frac{\operatorname{tr}(SL_T - S^{-1}L_T - SL_T^{-1} + (L_TS)^{-1})}{8 \sinh \frac{i\alpha}{2} \sinh \frac{\gamma_\alpha}{2}} \\
&= \frac{2(\operatorname{tr}(SL_T) - \operatorname{tr}(S^{-1}L_T))}{8 \sinh \frac{i\alpha}{2} \sinh \frac{\gamma_\alpha}{2}} \\
&= \frac{\operatorname{tr}(S)\operatorname{tr}(L_T) - 2\operatorname{tr}(S^{-1}L_T)}{4 \sinh \frac{i\alpha}{2} \sinh \frac{\gamma_\alpha}{2}} \\
&= \frac{\operatorname{tr}(S)\operatorname{tr}(L_T) - 2\operatorname{tr}(S)}{4 \sinh \frac{i\alpha}{2} \sinh \frac{\gamma_\alpha}{2}} \\
&= \frac{\operatorname{tr}(S)(\operatorname{tr}(L_T) - 2)}{4 \sinh \frac{i\alpha}{2} \sinh \frac{\gamma_\alpha}{2}} \\
&= \frac{2 \cos \frac{\alpha}{2} (2 \cosh \frac{\gamma_\alpha}{2} - 2)}{4i \sin \frac{\alpha}{2} \sinh \frac{\gamma_\alpha}{2}} \\
&= -i \cot \frac{\alpha}{2} \tanh\left(\frac{\gamma_\alpha}{4}\right).
\end{aligned}$$

where the first equality comes from [3, p. 68], the sixth equality comes from the Cayley-Hamilton theorem, and the seventh equality comes from Lemma 4.1. Therefore, we have

$$i \cosh \rho = \cot \frac{\alpha}{2} \coth\left(\frac{\gamma_\alpha}{4}\right).$$

□

Pythagorean theorem 4.2 gives the following theorem which relates the eigenvalues of $h(l)$ and $V = \cosh \rho$ for $A = \cot \frac{\alpha}{2}$.

Theorem 4.3. *Recall that l is the longitude. By conjugating if necessary, we may assume $h(l)$ is upper triangular. Let $L = h(l)_{11}$. Let $A = \cot \frac{\alpha}{2}$. Then the following formulae show that there is a one to one correspondence between the eigenvalues of $h(l)$ and $V = \cosh \rho$:*

$$iV = A \frac{L - 1}{L + 1} \text{ and } L = \frac{A - iV}{A + iV}.$$

Proof. By Theorem 4.2,

$$\begin{aligned}
iV &= i \cosh \rho \\
&= \cot \frac{\alpha}{2} \tanh\left(\frac{\gamma_\alpha}{4}\right) \\
&= \cot \frac{\alpha}{2} \frac{\sinh\left(\frac{\gamma_\alpha}{4}\right)}{\cosh\left(\frac{\gamma_\alpha}{4}\right)} \\
&= \cot \frac{\alpha}{2} \frac{e^{\frac{\gamma_\alpha}{4}} - e^{-\frac{\gamma_\alpha}{4}}}{e^{\frac{\gamma_\alpha}{4}} + e^{-\frac{\gamma_\alpha}{4}}} \\
&= \cot \frac{\alpha}{2} \frac{e^{\frac{\gamma_\alpha}{2}} - 1}{e^{\frac{\gamma_\alpha}{2}} + 1} \\
&= A \frac{L - 1}{L + 1}.
\end{aligned}$$

If we solve the above equation,

$$iV = A \frac{L - 1}{L + 1},$$

for L , we have

$$L = \frac{A - iV}{A + iV}.$$

□

5. PROOF OF THEOREM 1.1

According to [19, 8, 10, 20], there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ such that $X(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi]$. Denote by $D(X(\alpha))$ the discriminant of $P(V, A)$ over V . Then α_0 is the only zero of $D(X(\alpha))$ in $[\frac{2\pi}{3}, \pi)$.

From Theorem 4.3, we have the following equality,

$$(1) \quad |L|^2 = \left| \frac{A - iV}{A + iV} \right|^2 = \frac{|A|^2 + |V|^2 + 2A \operatorname{Im} V}{|A|^2 + |V|^2 - 2A \operatorname{Im} V}.$$

For the volume, we choose L with $|L| \geq 1$ and hence we have $\operatorname{Im}(V) \geq 0$ by Equality (1). The component of V with $\operatorname{Im}(V) \geq 0$ which becomes real at α_0 has negative real part. On the geometric component which gives the unique hyperbolic structure, we have the

volume of a hyperbolic cone-manifold $X(\alpha)$ for $0 \leq \alpha < \alpha_0$:

$$\begin{aligned}
\text{Vol}(X(\alpha)) &= - \int_{\alpha_0}^{\alpha} 2 \left(\frac{l_{\alpha}}{2} \right) d\alpha \\
&= - \int_{\alpha_0}^{\alpha} 2 \log |L| d\alpha \\
&= - \int_{\pi}^{\alpha} 2 \log |L| d\alpha \\
&= \int_{\alpha}^{\pi} 2 \log |L| d\alpha \\
&= \int_{\alpha}^{\pi} 2 \log \left| \frac{A - iV}{A + iV} \right| d\alpha,
\end{aligned}$$

where the first equality comes from the Schläfli formula for cone-manifolds (Theorem 3.20 of [1]), the second equality comes from the fact that $l_{\alpha} = |Re(\gamma_{\alpha})|$ is the real length of the one longitude of $X(\alpha)$, the third equality comes from the fact that $\log |L| = 0$ for $\alpha_0 < \alpha \leq \pi$ by Equality (1) since all V 's are real for $\alpha_0 < \alpha \leq \pi$, and $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ is the zero of the discriminant $D(X(\alpha))$. Numerical calculations give us the following value for α_0 : $\alpha_0 \approx 2.83003$.

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